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Magnetic fields and Brownian motion on the 2-sphere

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Abstract. Using constrained path integrals, we study some statistical properties of Brownian paths on the two-dimensional sphere. A generalized Lévy law for the probability $\mathscr{P}(A)$ that a closed Brownian path encloses an algebraic area A is obtained. Distributions of scaled variables related to the winding of paths around some fixed point are recovered in the asymptotic regime $t \to \infty$.

The study of the rotational Brownian motion goes back originally to the works of Perrin [1] and Lévy [2]. It gave birth to the study of stochastic processes over Lie groups which now stands as an autonomous branch of stochastic processes [3]. An interesting physical realization which can be cast in this framework is the motion of a spin in a random magnetic field [4]. The work presented here is a continuation of a recent work [5] in which some properties of the winding of the planar Brownian motion were studied. Following Edward's observation [6], one can describe a constrained Brownian process, in the path integral framework, in terms of spectral properties of some Hamiltonians coupled to magnetic fields [7]. Path integral techniques also permit one to access other quantities, like e.g. the area between a Brownian path and its subtending chord [8]. Similar methods have been recently used in the literature, especially in the study of localization of the interface line in the 2D Ising model [9].

In this paper, we are concerned with diffusion processes on the sphere. We determine the probability distribution $\mathscr{P}(A)$ of the area A enclosed by a closed Brownian path that winds on the sphere, thus providing a generalization of Lévy's formula [10]. The method we use also allows us to reach some limiting laws, and, especially, the law that gives the asymptotic behaviour of a Brownian path which winds around the pole's axis. If $\phi(t)$ stands for a continuous determination of the azimuthal angle that the particle has wound at time t, one can show, for $t \to \infty$, that the scaled variable $\phi(t)/t$ is distributed according to a Cauchy law. Although this result was already known in the mathematical literature [11], we believe that it is interesting to provide a heuristic derivation. Indeed, our approach shows in a very clear way the relation between the existence of a scaling variable, in the long time asymptotic regime, and the spectral properties at low energy of some quantum mechanical Schrödinger operators. A more systematic study of this relationship will be presented elsewhere [12].

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Let us consider the two-dimensional sphere S^2 of radius R as an embedded manifold in R^3. If $\boldsymbol{n}$ is an outward unit vector perpendicular to the sphere $S^2=\{\boldsymbol{r}/|\boldsymbol{r}|=R\}$, one can define on the punctured sphere $S^2\backslash\{P\}$ the vector field

$$\boldsymbol{V}_n(\boldsymbol{r})=R^2\frac{\boldsymbol{n}\wedge\boldsymbol{r}}{r(r+\boldsymbol{r}\cdot\boldsymbol{n})} \tag{1}$$

P being the point such that $\boldsymbol{r}=-R\boldsymbol{n}$. Then, with every oriented, closed path (L) such that $P\notin(L)$, one can associate a real number $A_n(L)$ given by the line integral

$$A_n(L)=\oint_{(L)}\boldsymbol{V}_n(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}. \tag{2}$$

For $\boldsymbol{n}$, $\boldsymbol{n}'$ being two arbitrary directions, one has

$$A_n(L)-A_{n'}(L)=4\pi R^2k \qquad k\in Z. \tag{3}$$

We will interpret the quantity defined by (2) as the *algebraic area* enclosed by the path (L). The freedom of choice of the vector $\boldsymbol{n}$ imposes that this quantity is only defined modulo the total area of S^2. This is a consequence of topological obstructions which do not allow one to define a 1-form A such that $\omega_{\mathrm{vol}}=\mathrm{d}A$ on S^2. In the following we will choose a determination of the area such that $-2\pi R^2<A<2\pi R^2$.

The path functional

$$F[(L)]\equiv\sum_{k\in Z}\delta(A_n(L)-A+4\pi R^2k) \tag{4}$$

gives zero weight to all the paths (L) which do not enclose a given algebraic area A. Then, the number of configurations of area A (mod $4\pi R^2$) is obtained by constraining the Wiener measure over Brownian paths by the functional (4). For a closed path $\{\boldsymbol{r}(\tau)/\tau\in[0,t];\ \boldsymbol{r}_0=\boldsymbol{r}(0)=\boldsymbol{r}(t)\}$ travelled in time t, this number of configurations is given by

$$\mathcal{N}(A|\boldsymbol{r}_0]=N\int_{\boldsymbol{r}(0)=\boldsymbol{r}_0}^{\boldsymbol{r}(t)=\boldsymbol{r}_0}\mathscr{D}\boldsymbol{r}(\tau)F[\boldsymbol{r}(\tau)]\exp\left(-\frac{1}{2}\int_0^t\mathrm{d}\tau\,\dot{\boldsymbol{r}}^2(\tau)\right). \tag{5}$$

We have chosen units for which the diffusion constant $D=1$. N is a normalization factor. From this expression, an application of the Poisson sum formula leads to

$$\mathcal{N}(A|\boldsymbol{r}_0]=\frac{N}{4\pi R^2}\sum_{k\in Z}\mathrm{e}^{-\mathrm{i}kA/2R^2}\int_{\boldsymbol{r}(0)=\boldsymbol{r}_0}^{\boldsymbol{r}(t)=\boldsymbol{r}_0}\mathscr{D}\boldsymbol{r}(\tau)\exp\left(-\int_0^t\mathrm{d}\tau\,L[\boldsymbol{r}(\tau)]\right). \tag{6}$$

The Lagrangian $L=\boldsymbol{r}^2/2-\mathrm{i}\boldsymbol{A}(\boldsymbol{r})\cdot\dot{\boldsymbol{r}}$ describes the dynamics on $S^2\backslash\{P\}$ of a particle of unit mass and unit electric charge. It is coupled to the vector potential

$$\boldsymbol{A}(\boldsymbol{r})=\frac{k}{2}\frac{\boldsymbol{n}\wedge\boldsymbol{r}}{r(r+\boldsymbol{r}\cdot\boldsymbol{n})} \tag{7}$$

of a magnetic monopole of magnetic charge $k/2$ lying at the centre of the punctured sphere. We shall forget the unphysical singularity at P and consider that this object generates on S^2 a uniform magnetic field, because our convention for the algebraic area implies that Dirac's quantization condition holds [13].

In order to evaluate the total number of configurations $\mathcal{N}(A)$ of a given algebraic area A, one has to integrate (6) with the natural Riemannian measure on S^2 over all possible initial points. One gets that

$$\mathcal{N}(A)=\int_{S^2}\mathrm{d}^2\boldsymbol{r}_0\,\mathcal{N}(A|\boldsymbol{r}_0]=\frac{N}{4\pi R^2}\sum_{k\in Z}\exp\left(-\frac{\mathrm{i}kA}{2R^2}\right)Z_k \tag{8}$$

where $Z_k = \mathrm{Tr}\, \mathrm{e}^{-tH_k}$ is the partition function of a charged particle in the field of a magnetic monopole of strength $k/2$. The corresponding quantum mechanical rotational levels, obtained by Tamm in 1931 [14],

$$E_k = \frac{1}{2R^2}\left[\frac{|k|}{2}(2n+1)+n(n+1)\right] \qquad n \in N \tag{9}$$

have a degeneracy $d_k = |k|+2n+1$. Therefore, the probability $\mathcal{P}(A)$ that a closed Brownian path encloses an algebraic area A reads as a discrete Fourier series

$$\mathcal{P}(A) = \frac{\mathcal{N}(A)}{\int_{-2\pi R^2}^{2\pi R^2} \mathrm{d}A\, \mathcal{N}(A)} = \frac{1}{4\pi R^2}\sum_{k\in Z} \mathrm{e}^{-ikA/2R^2}\frac{Z_k}{Z_0}. \tag{10}$$

Here, the partition function Z_k is

$$Z_k = \sum_{n\in N}(|k|+2n+1)\exp\left(-\frac{t}{2R^2}\left[\frac{|k|}{2}(2n+1)+n(n+1)\right]\right). \tag{11}$$

With the help of the expansion [15]

$$\frac{\pi^2}{8}\frac{1}{\cosh^2(\pi x/2)} = \sum_{n\in N}\frac{(2n+1)^2-x^2}{[(2n+1)^2+x^2]^2} \tag{12}$$

an elementary calculation shows that, in the flat space limit $1/R^2 \to 0$, this probability behaves like

$$\mathcal{P}(A) \sim \frac{\pi}{2t}\frac{1}{\cosh^2(\pi A/t)}. \tag{13}$$

Thus, we recover the Lévy's formula for the usual flat Euclidean plane [10]. Equivalently, this can be interpreted by saying that for very short time, $t \to 0$, the Brownian particle only explores a small part of the sphere. Therefore, it does not feel the curvature for fixed $A \ll R^2$. This means that the short-time behaviour of Brownian diffusion is essentially independent of the metric properties as long as one considers smooth manifolds. For instance, one can consider the non compact hyperbolic plane of constant negative curvature $-1/R^2$. With the results of [16] for the heat kernel of the Laplace–Beltrami operator minimally coupled to a uniform magnetic field, it is easy to show along the same lines that Lévy's flat space limiting law for $\mathcal{P}(A)$ is again recovered.

On the contrary, one can expect that at the large times, the diffusion process will be mainly influenced by topological and/or metric properties of the space. For instance, on a smooth compact 2D manifold like S^2, there will exist an asymptotic equilibrium distribution of the diffusion equation $\partial P/\partial t - \Delta P = 0$, namely the constant distribution, equal to the inverse of the total volume of the manifold [see 17]. On the other hand, free Brownian motion on a non-compact manifold of infinite volume will certainly be dominated by metric effects as $t \to \infty$. In the second part of this paper, we therefore concentrate on this limit of the Brownian motion. More specifically, we shall be interested in the asymptotic behaviour of the azimuthal angle made by the particle with respect to the S^2-poles axis, and the equivalent problem on the infinite flat space†.

† Note that the related question for the hyperbolic plane is not really pertinent since it is known that, asymptotically, the non-recurrent Brownian motion almost surely keeps a fixed direction (in polar coordinates with respect to the origin) [18]. Intuitively, winding around some fixed point demands significant recurrence of the Brownian motion.

First, in order to define in a non-ambiguous way the total angle $\phi(t)$, one must twice puncture the sphere, e.g. one considers the sphere with the two poles omitted. This turns S^2 topologically into a cylinder, which has the same homotopy group as the circle S^1. Using the usual spherical coordinates θ, ϕ on this punctured sphere, the functional

$$\delta\left[\phi - \int_0^t \mathrm{d}\tau\, \dot{\phi}(\tau)\right] \tag{14}$$

enables one to restrict the paths to those which start from $\boldsymbol{r}_0$ at time zero and end at $\boldsymbol{r}$ at time t. The total azimuthal angle $\phi(t)$ swept out around the polar axis is restricted to be ϕ. Inserting this constraint into the Wiener integral representation of the process on S^2, one is formally led to the quantum dynamics of a charged particle in the magnetic field of a vortex. This is described by the Hamiltonian (Laplace–Beltrami operator on S^2 minimally coupled to the vortex field)

$$\hat{H} = \frac{-1}{2R^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi} + \mathrm{i}\lambda\right)^2\right]. \tag{15}$$

An elementary calculation gives the characteristic function

$$E(\mathrm{e}^{\mathrm{i}\lambda\phi(t)}|\boldsymbol{r}_0) \tag{16}$$

in terms of the spectrum of the Hamiltonian which is defined by

$$\hat{H}\psi_n(\boldsymbol{r},\lambda) = E_n(\lambda)\psi_n(\boldsymbol{r},\lambda). \tag{17}$$

One gets that

$$E(\mathrm{e}^{\mathrm{i}\lambda\phi(t)}|\boldsymbol{r}_0) = \int_{S^2} \mathrm{d}^2\boldsymbol{r} \sum_n \psi_n(\boldsymbol{r},\lambda)\psi_n(\boldsymbol{r}_0,\lambda)\,\mathrm{e}^{-tE_n(\lambda)}. \tag{18}$$

Here, n is a generic set of indices labelling the eigenstates. As previously discussed, the existence and form of the long-time behaviour will be strongly influenced by the properties of the manifold. The above formula clearly shows that the long-time limit is intimately related to the bottom of the quantum spectrum. More precisely, a scaled variable $\phi(t)/f(t)$ is going to exist if

$$\lim_{t\to\infty} E(\mathrm{e}^{\mathrm{i}\lambda\phi(t)/f(t)}) \tag{19}$$

is independent of t. As we shall see, this property depends in a crucial way on the limiting behaviour of the energy levels as the vortex strength λ goes to zero. In order that the integral (18) does not vanish for large time, one must have that the

$$\lim_{t\to\infty} tE_n(\lambda/f(t)) \tag{20}$$

is independent of t.

For the sphere S^2, the spectrum of the charged particle in the vortex field is labelled by two quantum numbers m, k [19]

$$\psi_{m,k}(\boldsymbol{r},\lambda) = c_{m,k}(\lambda) P_{k+|m+\lambda|}^{-|m+\lambda|}(\cos\theta)\,\mathrm{e}^{im\phi} \tag{21a}$$

$$E_{m,k}(\lambda) = \frac{1}{2R^2}[k+|m+\lambda|][k+|m+\lambda|+1]. \tag{21b}$$

The integration over the final point $\boldsymbol{r}$ selects the states with $m=0$. In the asymptotic limit, the dominant contribution to the integral (18) comes from the ground state, $k=0$. ($E_{0,0}(\lambda)=|\lambda|(|\lambda|+1)/2R^2$.) Condition (20) determines the scaled variable $\phi(t)/t$. The characteristic function is, then, asymptotically given by

$$\lim_{t\to\infty} E(\mathrm{e}^{\mathrm{i}\lambda\phi(t)/t}) = \mathrm{e}^{-|\lambda|/2R^2}\int_{S^2} \mathrm{d}^2\boldsymbol{r}\,\psi_{0,0}(\boldsymbol{r},0)\psi_{0,0}(\boldsymbol{r}_0,0) = \mathrm{e}^{-|\lambda|/2R^2}. \tag{22}$$

We have taken into account the fact that the ground state of the perturbed Hamiltonian $\hat{H}_\lambda$ becomes constant as $\lambda\to 0$. By an inverse Fourier transform, we find that the scaled variable $\phi(t)/t$ is distributed according to a Cauchy law, i.e.

$$\mathscr{P}(x=\phi(t)/t)\,\mathrm{d}x \underset{t\to\infty}{\simeq} \frac{1}{\pi}\frac{2R^2\,\mathrm{d}x}{1+4R^4x^2}. \tag{23}$$

In contrast with the previous formula for $\mathscr{P}(A)$ (10), here one does not recover Spitzer's law for planar winding [20] when $R^2\to\infty$. According to the heuristic discussion on limiting behaviour of diffusion processes, this is not surprising. The topology of the compact sphere is very different from that of the plane. One has to consider a discrete spectrum of bound states in the former case, as opposed to a continuum of scattering states in the latter case.

Thus, in order to perform a similar analysis on the Euclidean plane, the formalism must be generalized to include continuum states. We start by using polar coordinates on the plane. A vortex of strength λ is localized at the origin. The corresponding quantum Hamiltonian

$$\hat{H}_\lambda = -\frac{1}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{1}{2r^2}\left(\frac{\partial}{\partial\phi}+\mathrm{i}\lambda\right)^2 \tag{24}$$

possesses a continuous spectrum on the positive real axis. The characteristic function can therefore be written in terms of Bessel's function $\mathscr{J}_\mu(z)$ as

$$\begin{aligned} E(\mathrm{e}^{\mathrm{i}\lambda\phi(t)}|\boldsymbol{r}_0) &= \int_0^\infty \mathrm{d}r\,r \int_0^\infty \mathrm{d}k\,k\mathscr{J}_{|\lambda|}(kr)\mathscr{J}_{|\lambda|}(kr_0)\,\mathrm{e}^{-tk^2/2} \\ &= \int_0^\infty \mathrm{d}r\,r\int_0^\infty \mathrm{d}k\,k\mathscr{J}_{|\lambda|}(kr)\mathscr{J}_{|\lambda|}(kr_0/\sqrt{t})\,\mathrm{e}^{-k^2/2}. \end{aligned} \tag{25}$$

The long-time limit is now governed by the behaviour of the wavefunction at the origin. From the behaviour of the Bessel function as $\lambda\to 0$ and $kr_0/\sqrt{t}\to 0$, one recovers the scaled variable $\phi(t)/\log t$ [21] and the probability distribution associated with it.

$$\lim_{t\to\infty} E(\mathrm{e}^{\mathrm{i}\lambda\phi(t)/\log t}) = \mathrm{e}^{-|\lambda|/2}. \tag{26}$$

It follows that $x=2\phi(t)/\log t$ is distributed according to a Cauchy law. This is Spitzer's [20] original result

$$\mathscr{P}(x)\,\mathrm{d}x = \frac{1}{\pi}\frac{\mathrm{d}x}{1+x^2}. \tag{27}$$

In summary, our analysis has emphasized the formal link between stochastic processes and quantum mechanics. We have given a simple derivation of limiting laws for free Brownian motions on two-dimensional manifolds by using Wiener path integrals and the spectral properties of Laplace–Beltrami operators coupled to magnetic fields. The limiting behaviour obtained in the case of the sphere, and more generally in the case of any compact two-dimensional manifold, is, in fact, independent of the precise structure of the wavefunctions. The asymptotic limit $t \to \infty$ is entirely controlled by the low energy behaviour of the ground state energy $E(\lambda)$, i.e. $\lambda \to 0$. The latter quantity can be found perturbatively. The singularity of the perturbation expansion is reflected by the non-analytic behaviour of $E(\lambda)$ in the coupling constant λ^2. These features are quite general. In the case of processes with drift [12] they allow one to recover the Lévy stable laws as limiting laws.

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